II. "Note on an Erroneous Extension of Jacobi's Theorem." By Isaac Todhunter, M.A., F.R.S. Received October 25, 1872.

1. It is well known that Jacobi discovered the possibility of the relative equilibrium of a mass of homogeneous fluid which is in the form of an ellipsoid and rotates with uniform angular velocity round the least principal axis. A few days since, in reading over for the press a manuscript which had been written last year, I observed I had drawn attention to the circumstance that such relative equilibrium would be impossible if the ellipsoid rotated round any other straight line. Almost immediately afterwards I was accidentally glancing for the first time at the elaborate treatise on Mechanics published in 1870 by Dr. W. Schell, under the title of "Theorie der Bewegung und der Kräfte," and I noticed an account of Jacobi's theorem. Dr. Schell records that Jacobi was led to the discovery of his theorem by reason of an erroneous statement, made by Pontécoulant, that such a result was impossible; Jacobi undertook the inquiry, as he said, by virtue of a "certain spirit of contradiction to which he owed his most important discoveries" (see page 966 of Dr. Schell's volume). It should be remarked, however, that as to the error, Pontécoulant merely followed Lagrange.

2. I was much surprised to find that on the same page Dr. Schell made the following assertion:—"It has been lately shown by Dahlander that the relative equilibrium of the rotating ellipsoid will subsist even when the axis of rotation does not coincide with a principal axis of the ellipsoid." A reference is supplied to a memoir by Dahlander in Poggendorff's 'Annalen,' vol. cxxix. (1866) p. 443. Notwithstanding this combination of authority the assertion is incorrect, as I shall now show.

3. I assume that when a mass of fluid is rotating with uniform angular velocity round a fixed axis, the problem of determining the pressure of the fluid and the form of the free surface may be changed from a dynamical form to a statical in the following manner:—Suppose the rotation stopped, and supply at every point an acceleration at right angles to the axis outwards from the axis, equal to $r\omega^2$, where ω is the angular velocity, and r is the distance of the point from the axis of rotation. This can be easily demonstrated, and is, in fact, always taken for granted by writers on the subject.

4. I will confine myself for simplicity to the case in which the assumed axis of rotation passes through the centre of the ellipsoid; but it will be easily seen that the process is applicable when this condition is not fulfilled. Suppose that an ellipsoid, of which the axes are $2a$, $2b$, $2c$, rotates about a diameter which makes with the principal axes angles whose direction cosines are l, m, n. Take for the equation to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let O denote the centre of the ellipsoid, M any point of the mass whose coordinates are x, y, z, and N the point where the perpendicular from M on the axis of rotation meets that axis. Let ξ, η, ζ be the coordinates of N.

Then the acceleration ω^2MN, when resolved parallel to the axes, gives rise to the components

$$\omega^2(x-\xi), \quad \omega^2(y-\eta), \quad \omega^2(z-\zeta)$$

respectively. We must now determine ξ, η, ζ.

We know that

$$\cos \text{MON} = \frac{lx+my+nz}{\text{OM}};$$

hence OM cos MON, that is ON$=lx+my+nz$. Denote this by v; then we have

$$\xi = lv, \quad \eta = mv, \quad \zeta = nv.$$

The attractions of the ellipsoid at (x, y, z) parallel to the axes will be respectively

$$\text{P}x, \quad \text{Q}y, \quad \text{R}z;$$

where P, Q, R are certain constants, in the form of definite integrals, which depend on the ratios of the axes of the ellipsoid.

Hence if p denote the pressure, and ρ the density of the fluid, we have

$$\frac{dp}{dx} = \rho\{\omega^2(x-lv) - \text{P}x\},$$

$$\frac{dp}{dy} = \rho\{\omega^2(y-mv) - \text{Q}y\},$$

$$\frac{dp}{dz} = \rho\{\omega^2(z-nv) - \text{R}z\}.$$

Therefore the form of the free surface will be determined by the equation

$$\omega^2(x^2+y^2+z^2) - \omega^2(lx+my+nz)^2 - \text{P}x^2 - \text{Q}y^2 - \text{R}z^2 = \text{constant}. \quad . \quad (2)$$

By supposition, (1) and (2) must represent the same surface. But this is obviously impossible, unless two out of the three l, m, n vanish. Thus the axis of rotation must coincide with one of the principal axes; and then it follows in the known way that this must be the *least* principal axis.

5. Now let us turn to the memoir by Dahlander in Poggendorff's 'Annalen.' The process is this. Dahlander supposes that there are three simultaneous angular velocities, ω, ω', ω'', round the axes of x, y, z respectively; and then he *assumes* the equation

$$\frac{dp}{\rho} = -(\text{P}-\omega'^2-\omega''^2)x\,dx - (\text{Q}-\omega^2-\omega'^2)y\,dy - (\text{R}-\omega'^2-\omega^2)z\,dz.$$

This equation, however, is unjustifiable. Dahlander does not say how he obtained it, so that it is impossible to point out exactly where his error lies. Perhaps the equation was supposed to hold in virtue of some unwarranted extension of the principle in article 3. To show that the equation is wrong, it is sufficient to observe that it makes $\frac{dp}{dx}$ involve only the

variable x; but the three angular velocities are equivalent to a single angular velocity, and then, as we see in article 4, we shall have $\frac{dp}{dx}$ involving also y and z.

6. The problem which Dahlander discusses is in reality much simpler than his enunciation implies; it amounts to this: investigate the conditions under which a fluid will be in equilibrium in the form of an ellipsoid, when, besides the attraction of the fluid, there are forces parallel to the principal axes, which may be denoted by fx, gy, hz respectively. Traces of such a problem appear in other places—as, for example, in Lagrange's 'Mécanique Analytique,' première partie, Sect. VII. This is, however, different from the problem of *rotating* fluid, which it was proposed to discuss.

7. There is nothing to call for remark in the mathematical work of the memoir, except that a wrong value is assigned to the definite integral $\int_0^1 \frac{(1-u^2)u^2du}{(1+\lambda^2u^2)^{\frac{3}{2}}}$. The correct value is

$$\frac{3+2\lambda^2}{2\lambda^5}\log\{\lambda+\sqrt{(1+\lambda^2)}\}-\frac{3\sqrt{(1+\lambda^2)}}{2\lambda^4}.$$

October 24, 1872.

III. Additional Note to the Paper "On a supposed Alteration in the Amount of Astronomical Aberration of Light produced by the Passage of the Light through a considerable thickness of Refracting Medium." By the PRESIDENT. Received November 2, 1872.

Some months since I communicated to the Royal Society the result of observations on γ Draconis made with the water-telescope of the Royal Observatory (constructed expressly for testing the equality of the coefficient of sidereal aberration, whether the tube of a telescope be filled with air, as usual, or with water) in the spring and autumn of 1871. Similar observations have been made in the spring and autumn of 1872, and I now place before the Society the collected results. It will be remembered, from the explanation in the former paper, that the uniformity of results for the latitude of station necessarily proves the correctness of the coefficient of aberration employed in the Nautical Almanac.

Apparent Latitude of Station.

		°	′	″
1871.	Spring	51	28	34·4
	Autumn	51	28	33·6
1872.	Spring	51	28	33·6
	Autumn	51	28	33·8

I now propose, when the risk of frost shall have passed away, to reverify the scale of the micrometer, and then to dismount the instrument.